On the stability of local meshless methods for approximating time-dependent problems

Žiga Vaupotič

Scientific Computing Laboratory, Jožef Stefan Institute, Ljubljana, Slovenia

Scope

1. Stability of operators and C_0-semigroups
2. Stability of meshless collocation methods
3. Geometry approximation via discrete measures
4. Stabilisation operators and control-theoretic ideas

Continuous problem

Setting: Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with sufficiently smooth boundary, and let $L^2(\Omega)$ denote the classical Lebesgue space.

Problem: Consider the linear advection–diffusion equation:

$$\begin{cases} u'(t) = \mathscr{A} u(t), \\ \mathscr{A} u := \varepsilon \Delta u + \boldsymbol{\beta} \cdot \nabla u, \\ u(0) \in D(\mathscr{A}) \end{cases}$$

where $\boldsymbol{\beta} \in C^1(\overline{\Omega}, \mathbb{R}^2)$ and Dirichlet data is imposed, so that $D(\mathscr{A}) = H_0^1(\Omega) \cap H^2(\Omega)$.

Recall that: the operator $\mathscr{A}$ generates a strongly continuous semigroup $T(t) = e^{t\mathscr{A}}$ on $L^2(\Omega)$.

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Stability of continuous problem

Question: What exponential growth bound does the semigroup generated by $\mathscr{A}$ satisfy?

The energy identity after integration by parts reads

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2(\Omega)}^2 + \varepsilon\|\nabla u(t)\|_{L^2(\Omega)}^2 + \frac{1}{2}\int_\Omega (\nabla \cdot \boldsymbol{\beta})|u(t)|^2\, dx = 0.$$

Hence, if $\nabla \cdot \boldsymbol{\beta} \geq 0$ in Ω, then

$$\frac{d}{dt}\|u(t)\|_{L^2(\Omega)}^2 \leq 0.$$

Thus $\|T(t)u_0\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)}$ for all $t \geq 0$ (semigroup is contractive).

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Problem formulation

Discrete geometry: Consider a family of quasi-uniform and equidistributed point clouds $X_h \subset \Omega$. Let $\pi_h : Y \to \mathbb{R}^{N(h)}$ be a sampling operator over a point cloud $X_h \subset \Omega$.

Discrete problem: Consider a family of differentiation matrices $\{\boldsymbol{A}_h\}_{h>0}$ over which we seek the approximation to the semigroup, that is $\boldsymbol{u}_h : (0,T] \to \mathbb{R}^{N(h)}$ that solves

$$\boldsymbol{u}_h'(t) = \boldsymbol{A}_h \boldsymbol{u}_h, \qquad \boldsymbol{A}_h = \varepsilon \boldsymbol{D}_h^{\triangle} + \operatorname{diag}(\boldsymbol{\beta}_x)\boldsymbol{D}_h^x + \operatorname{diag}(\boldsymbol{\beta}_y)\boldsymbol{D}_h^y.$$

Assume that the differentiation weights are computed with RBF-FD (poly + mon).

Discrete norms: To study convergence, we equip the spaces of nodal values $\mathbb{R}^{N(h)}$ with the sequence of norms

$$\|\boldsymbol{u}\|_{\ell^2(X_h)}^2 = \frac{|\Omega|}{N} \sum_{i=1}^{N(h)} (\boldsymbol{u})_i^2$$

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$$\|\boldsymbol{u}\|^2_{\ell^2(X_h)} = \frac{|\Omega|}{N} \sum_{i=1}^{N(h)} (\boldsymbol{u})_i^2$$

Convergence

We now wish to study the convergence in the sense of

$$\|\pi_h e^{t\mathscr{A}} u_0 - e^{t\boldsymbol{A}_h} \pi_h u_0\|_{\ell^2(X_h)} \to 0$$

Theorem

Suppose that $u \in C^1((0,T], C^{\ell+1}(\overline{\Omega}))$ and that the following holds

1. *Consistency: $\|\boldsymbol{A}_h \pi_h u - \pi_h \mathscr{A} u\|_{\ell^2(X_h)} \leq Ch^\alpha$ for all $u \in C^{\ell+1}(\overline{\Omega})$*
2. *Stability: $(\boldsymbol{A}_h \boldsymbol{v}_h, \boldsymbol{v}_h)_{\ell^2(X_h)} \leq \mu \|\boldsymbol{v}_h\|^2_{\ell^2(X_h)}$ for all $\boldsymbol{v}_h \in \mathbb{R}^{N(h)}$ (μ indep. of h)*

Then the method converges with the order h^α.

Problem: how to ensure stability?

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Discrete semigroup stability

Stability can be studied through an energy identity:

$$\frac{d}{dt}\|\boldsymbol{u}_h\|^2_{\ell^2(X_h)} = 2(\boldsymbol{A}_h\boldsymbol{u}_h, \boldsymbol{u}_h)_{\ell^2(X_h)} \leq 2\mu\|\boldsymbol{u}_h\|^2_{\ell^2(X_h)}$$

Hence $\|\boldsymbol{u}_h(t)\|_{\ell^2(X_h)} \leq e^{\mu t}\|\boldsymbol{u}_h(0)\|_{\ell^2(X_h)}$.

Question: Is the constant μ uniform in h?

Proposition

The following statements are equivalent:

1. *The semi-discrete problem is uniformly stable.*
2. *There exists $\mu \in \mathbb{R}$, independent of h, such that $\operatorname{sym}(\boldsymbol{A}_h) \preceq \mu\boldsymbol{I}$.*
3. $\lambda_{\max}(\operatorname{sym}\boldsymbol{A}_h) \leq \mu$ *uniformly in h.*

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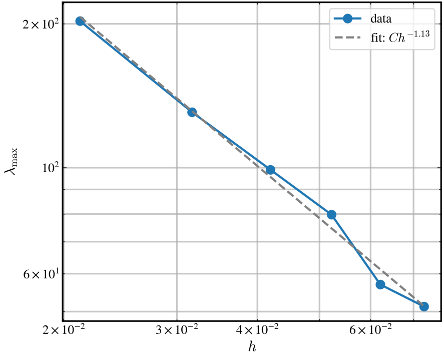

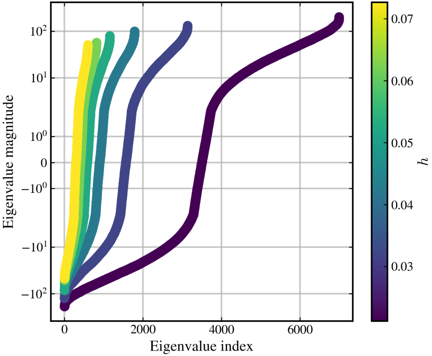

Estimates for stability

A direct inverse inequality gives, for a first-order operator,

$$(\boldsymbol{A}_h \boldsymbol{u}_h, \boldsymbol{u}_h)_{\ell^2(X_h)} \leq \|\boldsymbol{A}_h \boldsymbol{u}_h\|_{\ell^2(X_h)} \|u_h\|_{\ell^2(X_h)} \leq C h^{-1} \|\boldsymbol{u}_h\|^2_{\ell^2(X_h)}$$

The resulting growth bound is $O(h^{-1})$, so it is not uniform in h and does not provide the stability required for convergence. For a second-order operator, the corresponding bound is even less favourable.

Problem: Estimates for $\lambda_{\max}(\operatorname{sym} \boldsymbol{A}_h)$ are typically hard because geometric control does not directly yield the required control of the differentiation weights.

Recent results: T. Hangelbroek et. al (2025) studied spectral stability on a sphere.

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Overcoming stability issues

Oversampled RBF-FD methods: introduce a reconstruction to piecewise continuous functions $\mathscr{R}_h : \mathbb{R}^{N(h)} \to L^2(\Omega)$ and sample the PDE over $M \gg N$ points.

Result: Over the new pointcloud Y_h we obtain a discrete measure $\nu_M \to \mu$ that is uniformly convergent on $\mathscr{R}_h(\mathbb{R}^{N(h)})$ in M. Hence we can control

$$(\boldsymbol{A}_h \boldsymbol{u}_h, \boldsymbol{u}_h)_{\ell^2(Y_h)} - (\mathscr{A}_h u_h, u_h)_{L^2(\Omega)}$$

Problems: We must invert a mass matrix $\boldsymbol{M}_h$, since the problem is now

$$\boldsymbol{M}_h \boldsymbol{u}_h' = \boldsymbol{A}_h \boldsymbol{u}_h$$

The oversampling required for stability may scale poorly. Since the methods are local, the reconstruction is typically assembled from nonconforming local interpolation systems, making it difficult to control inconformities.

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Summation by parts: Construct a sequence of inner product spaces $(\mathbb{R}^{N(h)}, (\cdot,\cdot)_h)$, where

- the induced norms are uniformly equivalent to the discrete $\ell^2(X_h)$-norm.
- the discrete differential operators (*form a graded differential complex and*) satisfy a summation-by-parts formula, eg:

$$(\boldsymbol{D}_h \boldsymbol{u}_h, \boldsymbol{v}_h)_h + (\boldsymbol{u}_h, \boldsymbol{D}_h \boldsymbol{v}_h)_h = (\boldsymbol{u}_h, \boldsymbol{v}_h)_{\partial,h}.$$

Problems: This construction is difficult to implement for local collocation methods and requires substantial optimisation theory. Establishing uniform equivalence with $\ell^2(X_h)$ remains technically challenging.

Recent results: J. Glaubitz et al. (2024, 2026) - SBP operators on general function spaces (FSBP); N. Trask et al. (2016, 2019) - differential graded Frobenius algebras over graphs; J. Hicken et al. (2025) - pseudo-meshless polynomial constructions.

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Penalty operators: by constructing a penalty matrix $\boldsymbol{P}_h$, we obtain a perturbed semidiscrete problem

$$[\boldsymbol{u}_h^\gamma]'(t) = \boldsymbol{A}_h \boldsymbol{u}_h^\gamma + \gamma_h \boldsymbol{P}_h \boldsymbol{u}_h^\gamma$$

Choosing γ_h so that $\mathrm{sym}(\boldsymbol{A}_h + \gamma_h \boldsymbol{P}_h) \preceq \mu I$ uniformly gives stability. If the penalty is also consistent, this leads to

$$\|\boldsymbol{u}_h^\gamma - \pi_h u\|_{\ell^2(X_h)} \to 0.$$

Some operators: hyperviscosity $-\Delta^2$, symmetric hyperviscosity $-\Delta^T \Delta$ (in ℓ^2 sense), jump penalty over a reconstruction, graph penalty

Recent studies: V. Shankar et. al (2018) - "robust" hyperviscosity on manifolds, I. Tominec (2023, 2025) - symmetric hyperviscosity and jump penalty, M. Rot et al. (2026)...

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Overcoming stability issues

Penalty operators: by constructing a penalty matrix $\boldsymbol{P}_h$, we obtain a perturbed semidiscrete problem

$$[\boldsymbol{u}_h^\gamma]'(t) = \boldsymbol{A}_h \boldsymbol{u}_h^\gamma + \gamma_h \boldsymbol{P}_h \boldsymbol{u}_h^\gamma$$

Choosing γ_h so that $\mathrm{sym}(\boldsymbol{A}_h + \gamma_h \boldsymbol{P}_h) \preceq \mu I$ uniformly gives stability. If the penalty is also consistent, this leads to

$$\|\boldsymbol{u}_h^\gamma - \pi_h u\|_{\ell^2(X_h)} \to 0.$$

Some operators: hyperviscosity $-\Delta^2$, symmetric hyperviscosity $-\Delta^T \Delta$ (in ℓ^2 sense), jump penalty over a reconstruction, graph penalty

Recent studies: V. Shankar et. al (2018) - "robust" hyperviscosity on manifolds, I. Tominec (2023, 2025) - symmetric hyperviscosity and jump penalty, M. Rot et al. (2026)...

Theorem

Let $u \in C^1([0,T], C^{\ell+1}(\overline{\Omega}))$ solve the exact problem and let the scheme be pth-order accurate. Suppose that γ is chosen such that $\mathrm{sym}(\boldsymbol{A}_h + \gamma_h \boldsymbol{P}_h) \preceq \mu I$. Then the error of the perturbed problem satisfies

$$\|\boldsymbol{u}_h^\gamma(t) - \pi_h u(t)\|_{\ell^2(X_h)} \leq C' h^p \frac{e^{\mu t} - 1}{\mu} + \mathcal{E}_h(t).$$

where C is independent of h and $\mathcal{E}_h$ is the stabilisation error defined as

$$\mathcal{E}_h(t) = \int_0^t e^{\mu(t-s)} \|\gamma_h \boldsymbol{P}_h \pi_h u(s)\|_{\ell^2(X_h)}\, ds.$$

Result: Stability is not enough: convergence also requires control of $\|\gamma_h \boldsymbol{P}_h \pi_h u(s)\|_{\ell^2(X_h)}$.

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Penalty: Stability

How to choose stability constant γ_h?

Restriction on the penalty: negative semidefiniteness is a practical requirement, and its kernel must be sufficiently small.

We can pose the search for γ_h as a semidefinite program:

$$\begin{cases} \min_\gamma & \gamma \\ \text{s.t.} & \operatorname{sym}(\boldsymbol{A}_h - \mu \boldsymbol{I} + \gamma \boldsymbol{P}_h) \preceq 0, \\ & \gamma \geq 0 \end{cases}$$

Proposition

Let $\operatorname{sym} \boldsymbol{P}_h \preceq 0$. *Then there exists a finite* $\gamma_h \geq 0$ *such that* $\operatorname{sym}(\boldsymbol{A}_h - \mu \boldsymbol{I} + \gamma_h \boldsymbol{P}_h) \preceq 0$ *if and only if* $\operatorname{sym} \boldsymbol{A}_h - \mu \boldsymbol{I}|_{\ker(\operatorname{sym} \boldsymbol{P}_h)} \preceq 0$ *and* $\ker(\operatorname{sym} \boldsymbol{A}_h - \mu \boldsymbol{I}|_{\ker(\operatorname{sym} \boldsymbol{P}_h)}) \subseteq \ker(\operatorname{sym} \boldsymbol{A}_h - \mu \boldsymbol{I})$.

Interpretation: every mode invisible to the penalty must already be stable.

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Suppose that γ_h exists, with respect to the decomposition $\mathbb{R}^{N(h)} = \ker(\operatorname{sym} \boldsymbol{P}_h) \oplus \ker(\operatorname{sym} \boldsymbol{P}_h)^{\perp}$, write,

$$\operatorname{sym}(\boldsymbol{A}_h - \mu \boldsymbol{I}) = \begin{bmatrix} \boldsymbol{Q}_{00} & \boldsymbol{Q}_{01} \\ \boldsymbol{Q}_{01}^T & \boldsymbol{Q}_{11} \end{bmatrix}, \qquad -\operatorname{sym} \boldsymbol{P}_h = \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{P}_{11} \end{bmatrix},$$

where $\boldsymbol{P}_{11} \succ 0$. Then

$$\hat{\gamma}_h = \max\left\{0,\ \lambda_{\max}\left(\boldsymbol{Q}_{11} + \boldsymbol{Q}_{01}^T(-\boldsymbol{Q}_{00})^{\dagger}\boldsymbol{Q}_{01}, \boldsymbol{P}_{11}\right)\right\},$$

This leads to the following a priori scaling

$$\hat{\gamma}_h \leq \frac{\max\{0, \lambda_{\max}(\operatorname{sym} \boldsymbol{A}_h) - \mu\}}{\lambda_h^P}, \quad \lambda_h^P = \min\{\lambda > 0 : \lambda \in \sigma(-\operatorname{sym} \boldsymbol{P}_h)\}.$$

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Penalty

Result: The smallest nonzero eigenvalue of $-\operatorname{sym}\boldsymbol{P}_h$ gives a scaling estimate for the required penalty strength and, for sufficiently consistent penalties, a convergence result.

Problem: obtaining a priori estimates for λ_h^P is typically not feasible, so its scaling must often be studied numerically.

Differential operators such as $-\Delta^2$ are not automatically negative semidefinite by construction.

Algebraic operators, such as graph Laplacians, are therefore attractive. Proving consistency $\|\gamma_h \boldsymbol{P}_h \pi_h u(s)\|_{\ell^2(X_h)} \to 0$ remains a major obstacle.

Further research: finding an operator that satisfies both properties is important. A weighted graph Laplacian can serve as a key construction due to its generality.

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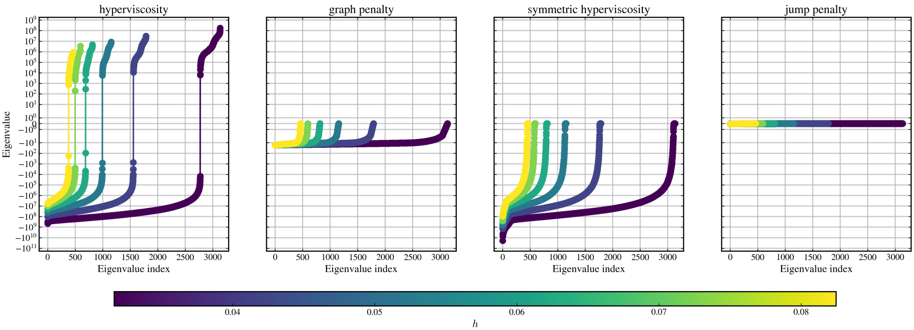

hyperviscosity
graph penalty
symmetric hyperviscosity
jump penalty
Eigenvalue
Eigenvalue index
0.04
0.05
0.06
0.07
0.08
h

Norms for convergence

1. We have a sequence of spaces $\mathbb{R}^{N(h)}$ without an intrinsic norm
2. If we were to choose a sequence of norms, how would we obtain a relevant sequence?

Idea: Mimic the continuous $L^2(\Omega, \mu)$ topology using discrete measures $\mu_h = \sum_i w_i \delta_{x_i}$ over point clouds $\{X_h\}_h$, with $\mu_h \rightharpoonup \mu$ in the weak* sense.

This leads to an isomorphism $L^2(X_h, \mu_h) \cong (\mathbb{R}^N, \|\cdot\|_{\ell^2(X_h)})$, where

$$\|\boldsymbol{u}\|_{\ell^2(X_h)} = \left(\sum_{i=1}^{N(h)} w_i |\boldsymbol{u}_i|^2 \right)^{1/2}.$$

For the quasi-uniform point clouds considered here, take $w_i = |\Omega|/N(h)$.

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Take-home messages

- Local consistency does not imply stability.
- Stability is governed by $\lambda_{\max}(\operatorname{sym} \boldsymbol{A}_h)$.
- A negative-semidefinite penalty should be used to control unstable modes outside its kernel.
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