Properties and benefits of using a hybrid RBF approximation for hyperviscosity stabilisation

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Scope

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General stability theory

- General stability of RBF-FD methods
- Global cardinal functions and stability

Stabilisation methods

- Hyperviscosity stabilisation
- Computational complexity of the hyperviscosity operator
- Hybrid schemes for computing the operator

- Discretisation in $Y = \{y_i\}_{i=1}^N$ evaluation/computational nodes
- Approximation u_h of the differential operator $\mathcal{L}$ applied to u

$$(\mathcal{L}u_h)(x) = \sum_{i=1}^{N} \mathcal{L}\Psi_i(x) u_h(y_i) \tag{1}$$

- Local interpolation with PHS r^k and monomials of order m for $x \in \mathcal{V}_y$

$$(\mathcal{L}u_h)(x) = \sum_{i=1}^{n} \mathcal{L}\psi_i^{(y)}(x) u_h(y_i) \tag{2}$$

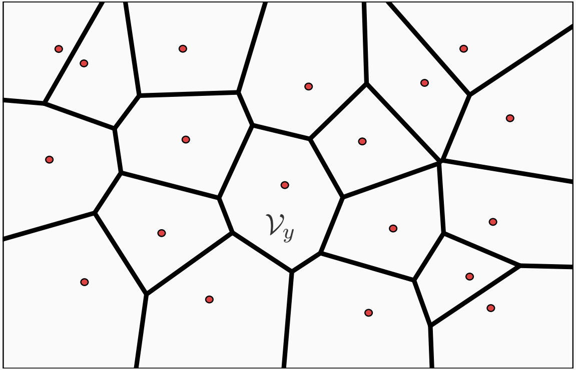


Fig. 1: Domain discretisation

- Global cardinal functions are piecewise functions - $\rho(x)$ returns the closest $y \in Y$

$$\Psi_i(x) = \begin{cases} \psi_j^{\rho(x)}(x), & x_i \in Y_{\rho(x)} \\ 0, & x_i \notin Y_{\rho(x)} \end{cases} \tag{3}$$

- We require the solution $u_h \in \tilde{V}_h$ in the pointwise sense, but can think of u_h continuously on the piecewise function space $V_h(\Omega)$ spanned by Ψ_i.
- Integration error - cause spurious growth under insufficient oversampling
- **Smoothing error**

- Global cardinal functions are discontinuous

$\iff$ RBF-FD trial space V_h is a discontinuous piecewise space *(solely piecewise continuous i.e.* $\Psi_i \in H^{k+1}(\mathcal{V}_y)$*)*

$\implies$ RBF-FD differentiation matrices may have spurious eigenvalues

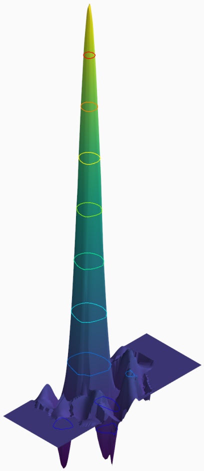


Fig. 2: Global cardinal function

General stability - stencil size

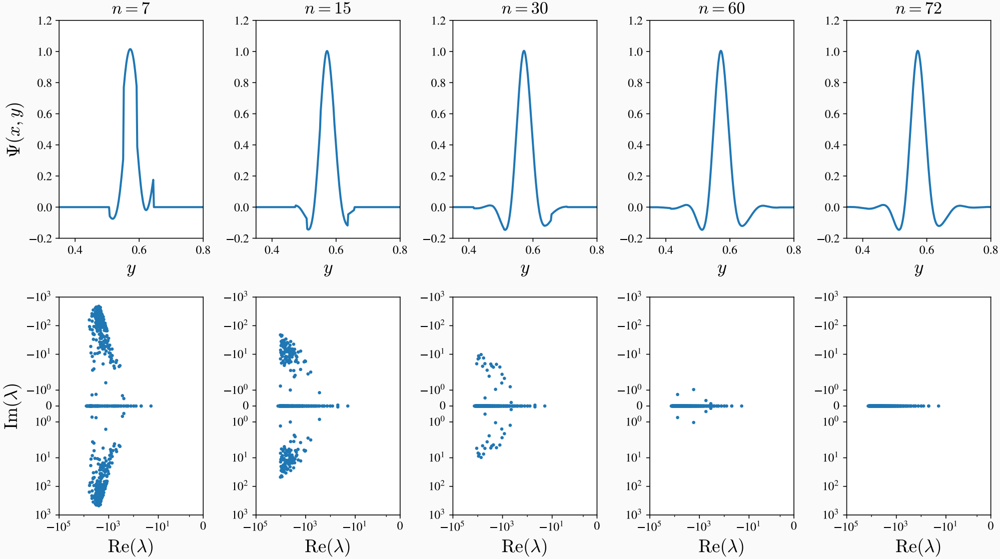
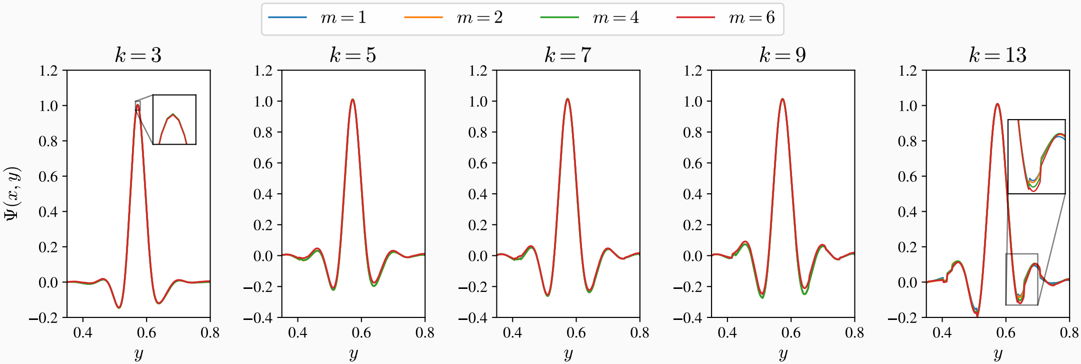


Fig. 3: Eigenspectra and global cardinal function Ψ^* of Laplace operator Δ

Fig. 4: Cardinal functions as a function of order k and m



Test case: Linear advection

$$\frac{\partial c}{\partial t} + u \cdot \nabla c = 0 \text{ on } \mathbb{T}^2([0,1])$$

$$c(x,0) = e^{-\frac{||x-c||^2}{2R^2}}$$

- Initially $c(x,0) \in C^\infty(\Omega)$, however, for every $t > 0$ we only have $c_h(x,t) \in V_h(\Omega) \subset L^2(\Omega)$, due to the projection onto $V_h(\Omega)$.
- Errors are multiplying under time stepping, particularly on the boundary of Voronoi regions, effectively also in the pointwise space.

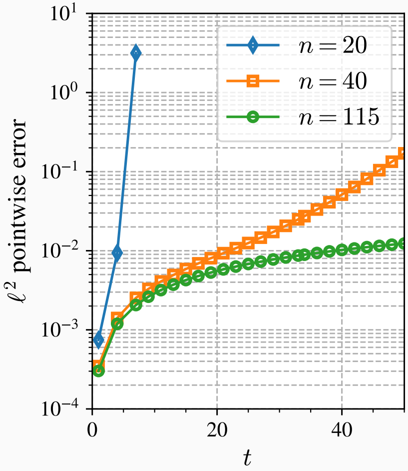


Fig. 5: Error as a function of time t

- As our $n \to N$ the jumps in cardinal functions are becoming smaller. Moreover, we have $V_h^{(n)} \subset H^{k+1}(\Omega)$ when $n = N$.

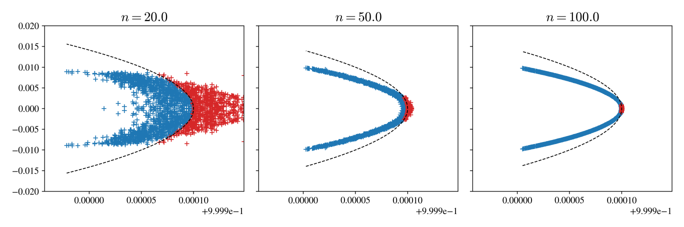


Fig. 6: Eigenspectrum as a function of stencil size n

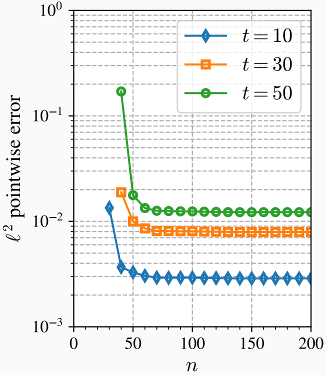


Fig. 7: Error as a function of stencil size n

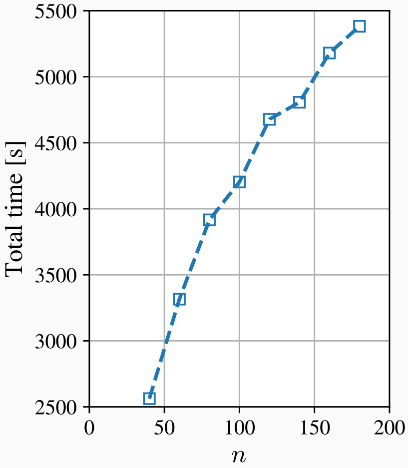


Fig. 8: Time for solving 10^5 iterations as a function of stencil size n

- **Undesirable:** Larger stencil sizes
- **Standard stabilisation:** Hyperviscosity stabilisation scheme
- **For hyperbolic equations:** Specialized jump penalty scheme

Example: Linear advection

$$\frac{\partial c}{\partial t} + u \cdot \nabla c = \gamma \Delta^\alpha c \qquad \gamma = (-1)^{\alpha+1} 2^{-2\alpha} h^{2\alpha}$$

- To smoothen the discontinuities caused by uneven local interpolation a high-order laplacian operator $\gamma\Delta^\alpha$ is added to the scheme
- The operator targets higher-order Fourier modes
- For first-order accuracy one requires $m = 2\alpha$
- **Problem:** Approximation of high-order derivatives requires large stencil sizes!

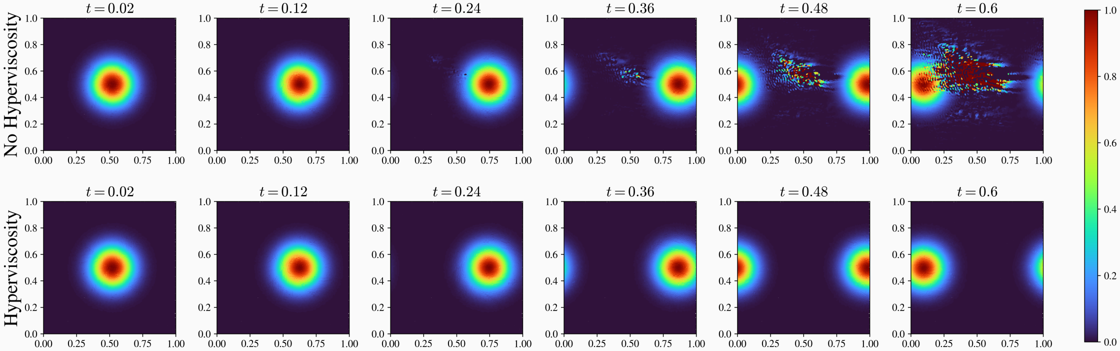


Fig. 9: Comparison of the scalar field c under stablisation and no stabilisation.

Hyperviscosity stabilisation: Monomials

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Different approximation schemes

$$\frac{\partial c}{\partial t} + u \cdot \nabla c = \gamma \tilde{\Delta}^\alpha c$$

- **Do we really need to have a first-order accurate operator?**
- The constant γ is $\mathcal{O}(h^{2\alpha})$
- We do not require a high precision of the operator, rather, that it targets high-order Fourier modes and the rest is controlled with γ
- To lower the stencil size, we can undersample the monomial term for approximating $\tilde{\Delta}^\alpha$
- **Cons:** unintended lower-order Fourier mode damping

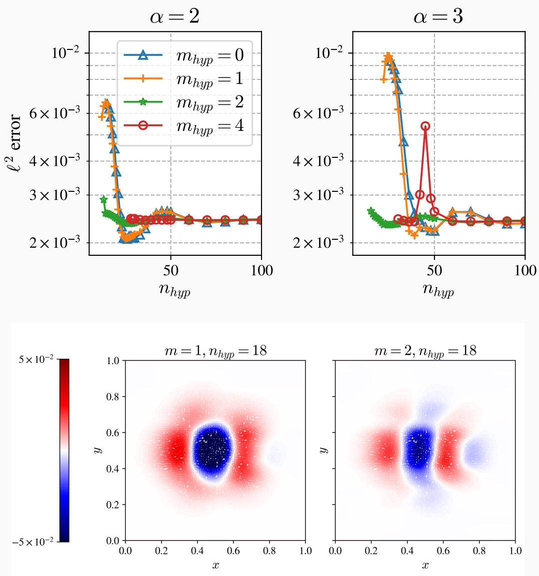

Fig. 10: Error as of hyperviscosity stencil size n_{hyp}

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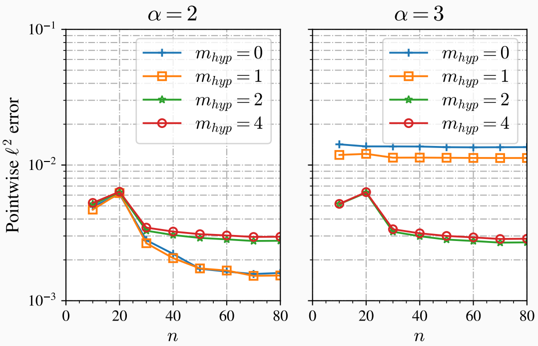

Fig. 11: Error as a function of stencil size n at $n_{hyp} = 35$

- **Interesting:** We can violate the local interpolation system unisolvency criteria $m = \lfloor \frac{k}{2} \rfloor - 1$ and stencil size recommendation.
- Under different severely undersampled monomials the error for approximating the operator is larger
- Different γ is required depending on the selection of the monomial term

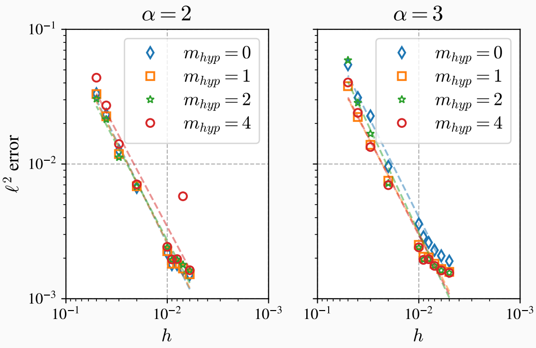

Fig. 12: Error as a function of h with $m = 2, n_{hyp} = 35$.

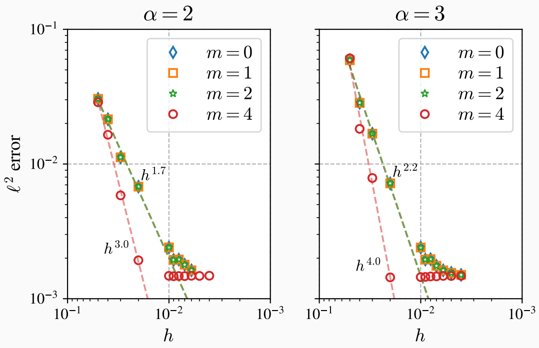

Fig. 13: Error as a function of h with $m_{hyp} = 2, n_{hyp} = 35$

- **Can we use other RBF based methods to compute the operator with a smaller stencil size?**
- Consider a local WLS approximation

$$u_h(x) = \sum_{i=1}^{m} \varphi_i^{(y)}(x) u_h(y_i) \qquad (4)$$

where $m < n$ and φ_i are Gaussian RBFs.

- The approach didn't work, since the shape parameter was difficult to select. The resulting approximation also caused low-order damping.

Presented:

- General stability depends on the stencil size
- Stabilisation via hyperviscosity also requires large stencil size
- The monomial augmentation of the RBF-FD scheme for hyperviscosity can be undersampled

Future work & discussion

- The constant γ is hard to select and depends on the approximation order
- The order of the hyperviscosity prodigiously affects the overall error, which is not expected

Thank you for your attention!

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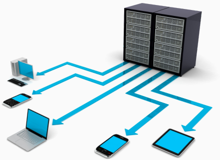

- When approximating the higher-order operator, there are huge round-off errors, since our weights w are $\mathcal{O}(h^{2\alpha})$