Oversampled RBF-FD Galerkin approach for solving elliptic problems

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Scope

1. RBF-FD and its extension to the oversampled approach
2. Oversampled method as a Galerkin method
3. Well-posedness and error estimates of the integral Galerkin method
4. Quasi-Monte Carlo quadrature and its analysis
5. Error estimates for quadrature scheme

Model problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with a Lipschitz boundary $\partial\Omega$. We seek $u : \Omega \to \mathbb{R}$ that satisfies (variationally)

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{1}$$

Assume that $f \in L^2(\Omega)$, *Calderon-Zygmund* theory states that $u \in V = H^2(\Omega) \cap H_0^1(\Omega)$ with

$$\|\Delta u\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)} \tag{2}$$

For numerical test, we consider $\Omega = [0, 1]^2$

$$f = -2\pi^2 \sin(\pi x) \sin(\pi y) \tag{3}$$

Motivation

Let $X = \{x_i\}_{i=1}^N \subset \Omega$ be a quasi-uniform nodeset with internodal distance of h. In the collocation method, we seek u_h such that

$$\Delta_h u_h(x_i) = f(x_i) \quad \forall i = 1, \dots N \tag{4}$$

This results in

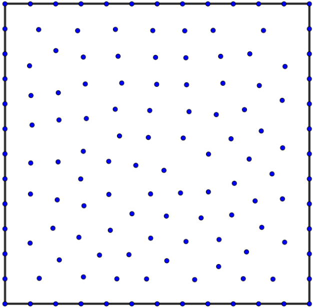
$$D_h u_h(X) = f(X) \tag{5}$$

How to prove that D_h is invertible? Is the problem well-posed? What can we state about the error?
For this, we consider the generalisation to the oversampled approach.

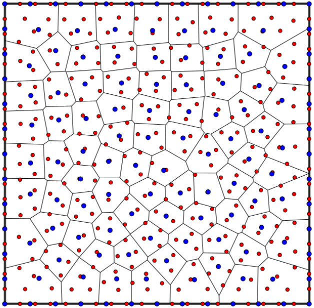
Oversampled RBF-FD approach

We introduce another low-discrepancy nodeset $Y = \{y_i\}_{i=1}^M$ where $M > N$. We define $M = qN$, assume that $q \in \mathbb{N}$. Let $\mathscr{K}_h = \{\mathscr{K}_i\}_{i=1}^N$ be a set of **shape-regular** Voronoi regions generated by X.

Discretisation $X \subset \Omega$



Sampling Y and K_h



Oversampled RBF-FD approach

For each $x_i \in X$ we generate the local interpolant (PHS + mon)

$$u_h|_{\mathscr{K}_i} = \sum_{k=1}^{n} \psi_k u_h(x_k) \tag{6}$$

Since DOFs are shared across interpolation systems we can construct global Lagrange functions and obtain

$$u_h = \sum_{i=1}^{N} \Psi_i u_h(x_i) \tag{7}$$

We require that it holds

$$\Delta u_h(y) = \sum_{i=1}^{N} u_h(x_i) \Delta \Psi_i(y) = f(y) \quad \forall y \in Y \tag{8}$$

Which results in

$$D_h(Y, X) u_h(X) = f(Y) \tag{9}$$

Oversampled RBF-FD approach

To solve this oversampled system, we project it onto the column space of the evaluation matrix $E_h(Y, X)$

$$E_h^T(Y, X) D_h(Y, X) u_h(X) = E_h^T(Y, X) f(Y), \tag{10}$$

where evaluation matrix maps $u_h(X) \mapsto u_h(Y)$. That is, the following diagram commutes

$$\begin{tikzcd}
V \arrow[r, "\mathcal{I}_h"] \arrow[d, "\pi_X"] & V_h \arrow[d, "\pi_Y"] \\
\mathbb{R}^N \arrow[r, "E_h"] & \mathbb{R}^M
\end{tikzcd}$$

This is equivalent to requiring that

$$D_h(Y, X) u_h(X) - f(Y) \perp \operatorname{span}\{E_h^{(1)}, \dots, E_h^{(N)}\} \tag{11}$$

Notice that column are just $E_h^{(i)} = [\Psi_i(y_1) \dots \Psi_i(y_M)]$

RBF-FD Trial space

The space $V_h = \mathrm{span}\{\Psi_i\}$ is an RBF-FD trial space, moreover for $k \geq 2$

$$V_h \subset H^k(\mathscr{K}_h) = \{u \in L^2(\Omega) : \, u|_{\mathscr{K}_i} \in H^k(\mathscr{K}_i)\} \tag{12}$$

We define the inner product over Y

$$(u_h, v_h) = \frac{|\Omega|}{M} \sum_{i=1}^{M} u_h(y_i) v_h(y_i)$$

Quadrature induced variational problem

Find $u_h \in V_h$ such that

$$(\Delta u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h \tag{13}$$

Which is an approximation of

$$\int_\Omega \Delta_h u_h \, v_h \, dx = \int_\Omega f v_h \, dx \quad \forall v_h \in V_h \tag{14}$$

Broken Sobolev space

We define a set of edges $\mathscr{E}_h$ between Voronoi regions. For each $e \in \mathscr{E}_h$ we have

$$[\![u]\!]_e = u^+ - u^- \quad \{\!\{u\}\!\}_e = \frac{u^+ + u^-}{2} \tag{15}$$

These values are well defined since for each Voronoi region there exists a trace operator $\gamma_i : H^k(\mathscr{K}_i) \to L^2(\partial \mathscr{K}_i)$.
Furthermore, the 0-trace space is $V_h^0 := \{u \in V_h : \gamma_i(u) = 0$ for all boundary $\mathscr{K}_i\}$. We let $V_h = V_h^0$ for clarity.

Integral Galerkin

Discrete variational problem

Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = -\int_\Omega \Delta_h u_h v_h \, dx = \int_\Omega f v_h \, dx = L(v_h) \quad \forall v_h \tag{16}$$

Note that $V_h \not\subset H_0^1(\Omega)$, we are commiting a variational crime.

Proposition (Consistency)

Let $a_h : (V + V_h) \times V_h \to \mathbb{R}$, *then* $u \in V$ *satisfies*

$$a_h(u, v_h) = L(v_h) \quad \forall v_h \tag{17}$$

This implies Galerkin orthogonality $a_h(u_h - u, v_h) = 0$ *for all* v_h.

Proof.

By density argument since $v \in C_c^\infty(\Omega)$ *and* $v_h \in L^2(\Omega)$. $\square$

Integral Galerkin

We define the following norm and seminorm on V_h

$$\|u_h\|_{V_h}^2 = \|\nabla_h u_h\|_{L^2(\Omega)}^2 + |u_h|_J^2 \quad |u_h|_J^2 = \sum_{e \in \mathscr{E}_h} \frac{1}{h_e} \|[\![u_h]\!]\|_{L^2(e)}^2 \tag{18}$$

Notice that the interpolant bound quickly follows

$$\|\mathcal{I}_h u - u\|_{V_h}^2 \leq C h^{2p+2} |u|_{W^{p+1,\infty}(\Omega)} + C' h^{2p+2} |u|_{W^{p+1,\infty}(\Omega)} \tag{19}$$

By using integration by parts and the standard identity $[\![\nabla_h u_h v_h]\!] = \{\!\{\nabla_h u_h\}\!\}[\![v_h]\!] + [\![\nabla_h u_h]\!]\{\!\{v_h\}\!\}$ we may rewrite the bilinear form as

$$\begin{aligned} a_h(u_h, v_h) = \int_\Omega \nabla_h u_h \nabla_h v_h \, dx &- \sum_{e \in \mathscr{E}_h} \int_e n_e \cdot \{\!\{\nabla_h u_h\}\!\}[\![v_h]\!] \, dx \\ &- \sum_{e \in \mathscr{E}_h} \int_e n_e \cdot [\![\nabla_h u_h]\!]\{\!\{v_h\}\!\} \, dx. \end{aligned}$$

Integral Galerkin

Proposition (Boundedness)

It holds that $|a_h(u_h, v_h)| \leq \|u_h\|_{V_h} \|v_h\|_{V_h}$.

Theorem (Coerecivity)

It holds that $a_h(u_h, u_h) \geq \alpha \|u_h\|_{V_h}^2$.

Proof.

Trace inequality + Young + interpolation weak continuity. □

Proposition (Error estimate)

For the solution $u_h \in V_h$ *it holds that*

$$\|u_h - u\|_{V_h} \leq C\|u - \mathcal{I}_h u\|_{V_h} \leq Ch^p \tag{20}$$

How can we find the L^2 estimate? Adjoint problem!

Integral Galerkin

Problem: method is not adjoint consistent! For $\psi \in V$ that satisfies

$$-\Delta\psi = g \quad \text{in } \Omega \quad \psi = 0 \quad \text{on } \partial\Omega \tag{21}$$

we want

$$\int_\Omega -\Delta v_h \psi \, dx = \int_\Omega g v_h \, dx \quad \forall v_h \in V_h + V \tag{22}$$

via integration by parts, we can show that

$$\int_\Omega -\Delta\psi v_h = \int_\Omega \psi(-\Delta_h v_h) \, dx + \sum_{e \in \mathscr{E}_h} \int_e [\![v_h \nabla\psi \cdot n]\!] \, dx \tag{23}$$

$$- \sum_{e \in \mathscr{E}_h} \int_e [\![\psi \nabla v_h \cdot n]\!] \, dx \tag{24}$$

This might reduce our convergence rate.

Integral Galerkin

We now solve

$$-\Delta\psi = u_h - u \quad \text{in } \Omega \quad \psi = 0 \quad \text{on } \partial\Omega \tag{25}$$

Defining $e_h = u_h - u$ we have that for $v_h = e_h$

$$\|e_h\|^2_{L^2(\Omega)} = a_h(\psi, e_h) + \text{Inconsistency terms} \tag{26}$$

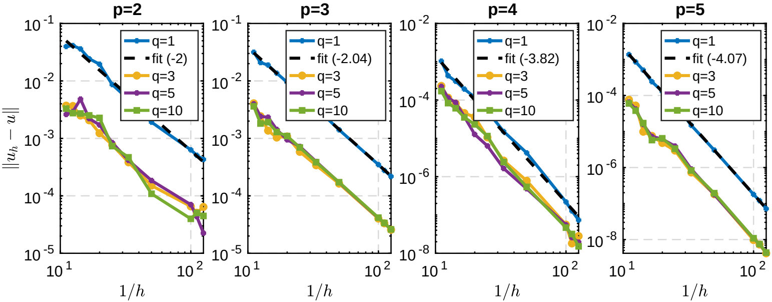
Using standard estimates in the sense that $a_h(\psi, e_h) = a_h(\psi - \mathcal{I}_h\psi, e_h)$ by galerkin orthogonality and by using boundedness we obtain

$$\|e_h\|^2_{L^2(\Omega)} = e_h e_h \, dx \leq \|\psi - \mathcal{I}_h\psi\|_{V_h} \|e_h\|_{V_h} + \text{Inconsistency terms} \tag{27}$$

notice that $\|\psi - \mathcal{I}_h\psi\| \leq h\|\Delta\psi\|_{L^2(\Omega)}$. By trace, elliptic and inverse inequalities, we get a final bound

$$\|e_h\|^2_{L^2(\Omega)} \leq h\|e_h\|_{V_h} + h^{-1}\|e_h\|_{V_h} \leq Ch^{p+1} + C'h^{p-1} \tag{28}$$

Error convergence



Quadarature

We define the quadrature operator $Q : V_h \to \mathbb{R}$ given by

$$Q_[f] = \frac{|\Omega|}{M} \sum_{i=1}^{M} f(y_i) \quad (u_h, v_h) = Q[u_h v_h] \tag{29}$$

Recall that we have the quadrature-induced variational problem

$$a_h^q(u_h, v_h) = (\Delta_h u_h, v_h) = (f, v_h) = L_h^q(v_h) \quad \forall v_h \in V_h \tag{30}$$

We have to show that the quadrature problem is coercive in V_h norm.

Integration error

We estimate the error for $u \in W^{1,1}_{mix}(\mathscr{K}_h)$ via Koksma-Hlawka inequality for QMC. That is, for $f \in BHKV([0,1]^d)$ and low discrepancy nodes Z

$$\left| N^{-1} \sum_{i=1}^{N} f(x_i) - \int_{[0,1]^d} f \, dx \right| \leq V(f) D(Z) \tag{31}$$

Consider a uniformity (and measure) preserving map $\varphi : \Omega \to [0,1]^2$ in the sense

$$\int_{\Omega} f(x) \, dx = |\Omega| \int_{[0,1]^2} f(\varphi(y)) \, dy = I[u_h] \tag{32}$$

Since $u \in W^{1,1}_{mix}(\mathscr{K}_h)$ we know that the Hardy-Kraus variation is bounded, i.e. $V(u) \leq \|\partial_x u\|_{L^1(\Omega)} + \|\partial_x u\|_{L^2(\Omega)} + \|\partial_{x,y} u\|_{L^1(\Omega)} < \infty$. Is $V(u \circ \varphi)$ bounded? Not necessarily. We **assume** that for $u \circ \varphi$

$$V(u \circ \varphi) \leq C V(u) \tag{33}$$

Preliminary results - Integration error

What about $u_h \in V_h$? Since $u_h \in H^2(\mathscr{K}_h)$, by the Sobolev embedding theorem (in the piecewise sense), $H^2(\mathscr{K}_h) \hookrightarrow W^{2,1}(\mathscr{K}_h) \hookrightarrow W^{1,1}_{mix}(\mathscr{K}_h)$. Hence

$$V(u_h) \leq \|u_h\|_{W^{1,1}_{mix}(\mathscr{K}_h)} \leq C\|u_h\|_{W^{2,1}(\mathscr{K}_h)} \leq C'h^{-2}\|u_h\|_{L^1(\Omega)} \tag{34}$$

it follows

$$|Q[u_h] - I[u_h]| \leq |\Omega|Ch^{-2}D(Y)\|u_h\|_{L^1(\mathscr{K}_h)} \tag{35}$$

Quadrature coerecivity / well-posedness

Following the derived inequality and applying Holder's inequality

$$
\begin{aligned}
a_h(u_h, u_h) - a_h^q(u_h, u_h) &\leq Ch^{-2} D(Y) \|\Delta_h u_h\|_{L^2(\Omega)} \|u_h\|_{L^2(\Omega)} && (36) \\
&\leq C' h^{-2} D(Y) |u_h|^2_{H^1(\mathscr{K}_h)} && (37)
\end{aligned}
$$

we used inverse inequality in the last step. It follows

$$
(a_h(u_h, u_h) - a_h^q(u_h, u_h)) \leq Ch^{-2} D(Y) \|u_h\|^2_{V_h} \tag{38}
$$

Hence

$$
\begin{aligned}
a_h^q(u_h, u_h) &= a_h(u_h, u_h) - (a_h(u_h, u_h) - a_h^q(u_h, u_h)) && (39) \\
&\geq \|u_h\|^2_{V_h} (\alpha - h^{-2} C D(Y)) && (40)
\end{aligned}
$$

Since $D(Y) \sim h_y^2 = q^{-1} h^2$, for sufficiently large q it follows that $(\alpha - Ch^{-2} D(Y)) > 0$.

Quadrature error estimate

We bound the overall error by Strang's second lemma

$$\|u-u_h\|_{V_h} \leq C\inf_{v_h}\|u-v_h\|_{V_h} + \sup_{w_h\in V_h\setminus\{0\}} \frac{|a_h^q(v_h,w_h)-L_h^q(v_h,w_h)|}{\|w_h\|_{V_h}}$$

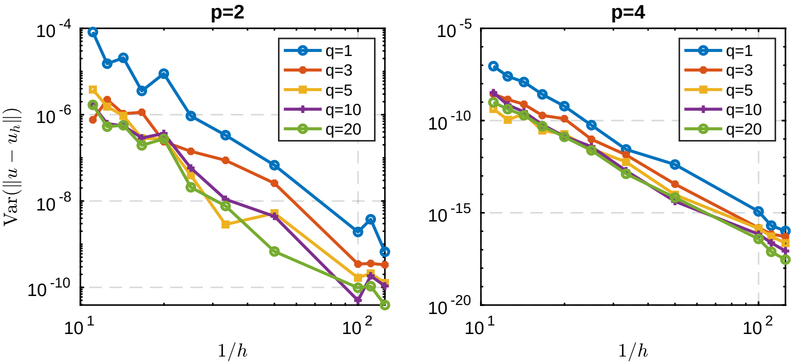
Recall that

$$\sup_{w_h\in V_h\setminus\{0\}} \frac{|a_h^q(v_h,w_h)-L_h^q(v_h,w_h)|}{\|w_h\|_{V_h}} = 0 \tag{41}$$

The obtained error in the energy norm

$$\|u-u_h\|_{V_h} \leq Ch^p \tag{42}$$

Numerical results



Thank you for your attention! Any questions?